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SOLUTION OF THE PROBLEM OF OPTIMAL CONTROL FOR HEATING OF A ONE-DIMENSIONAL PLATE AND ITS ASYMPTOTIC BEHAVIOR

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In recent years, many papers on optimization problems of heat and mass transfer [1–6] have been published. This interest is due to the importance of these problems for comprehension of heat- and mass-transfer processes and also in practical applications.

In the present paper, the classical problem of heating a one-dimensional plate is considered. One side of the plate is in contact with a well-mixed fluid layer. A constant temperature is maintained at the other side. The plate is heated by heat generation in the fluid layer (for example, by a heating element). Solutions of this problem under various initial and boundary conditions were considered by Carslaw and Jaeger [7].

We propose to approach this well-known problem from the viewpoint of optimal control. We have to find the optimal intensity of heat generation in a fluid that allows one to maximize the amount of heat accumulated in the plate at the moment of termination of the heating process. The natural constraints for the problem are the given amount of heat which can be generated in the fluid during the process, the fixed duration of the process, and the given minimum and maximum intensities of heat generation in the fluid. The Pontryagin maximum principle is used to solve it. As far as the author knows, the present paper is the first attempt to calculate the optimal intensity of heat generation in such a system.

Formulation of the Problem. Let us consider a one-dimensional plate one side of which ($x' = L$, where L is the thickness of the plate) is in contact with a well-mixed fluid (or a fluid of infinite thermal conductivity). Boundary conditions of the fourth kind, i.e., the equality of temperatures and heat fluxes, are assumed at the contact boundary. Let M_f denote the fluid mass per unit surface of the plate and c_f denote the specific heat of the fluid. Heat is generated in the fluid. The intensity of this generation per unit surface of the plate is a function of time $Q(t')$. It is assumed that the fluid releases heat only to the plate. In this case, the boundary condition for $x' = L$ can be written in the following form [7]:

$$k\frac{\partial T}{\partial x'} + M_f c_f \frac{\partial T}{\partial t'} = Q(t'),$$

where T is the temperature, k is the thermal conductivity of the plate, t' is the time, and x' is the linear coordinate.

The constant temperature T_0 is kept at the boundary $x' = 0$. The initial temperatures of the fluid and of the plate are also equal to T_0.

The solution of this problem follows from the results obtained in [7] and can be represented as

$$\theta(x,t) = 2H\sum_{n=1}^{\infty}\frac{\beta_n\sin(\beta_n x)\exp(-\beta_n^2 t)}{\cos\beta_n\{\beta_n^2 + H^2 + H\}}\int_0^t q(\tau)\exp(\beta_n^2\tau)\,d\tau. \tag{1}$$

Here β_n are the positive solutions of the transcendental equation

$$\beta_n \tan\beta_n = H. \tag{2}$$

The nondimensional parameters in Eqs. (1) and (2) are determined by the coordinate $x = x'/L$, the time $t = kt'/(\rho_s c_s L^2)$, the intensity of heat generation $q = QL/[k(T_1 - T_0)]$, the temperature $\theta = (T - T_0)/(T_1 - T_0)$,

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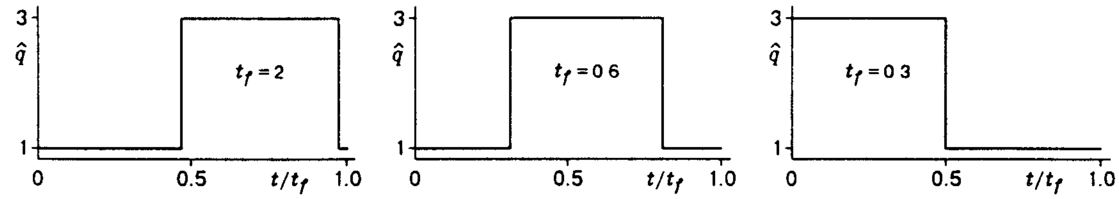

Fig. 1

and by the ratio of the heat-accumulating capacities of the plate and the fluid $H = \rho_s c_s L/(M_f c_f)$, where ρ_s is the plate density and T_1 is an arbitrary constant temperature used for normalization.

Let us consider the following optimization problem. The intensity of heat generation in the fluid which is assumed to be a piecewise continuous function of the time t and can vary from a minimum value $u_{\min}$ to a maximum value $u_{\max}$ is used as a control parameter. The minimum (maximum) value corresponds to the minimum (maximum) power of a heater. We seek to maximize the amount of heat accumulated in the plate by a given time, which corresponds to a process duration with the following constraints: the amount of heat which can be generated in the fluid and the duration of the process are fixed.

Mathematically, the problem considered is formulated as follows. One needs to find a function $\hat{q}(t)$ that maximizes the functional

$$\Phi(q) = \int_0^1 \theta(x, t_f)\, dx \to \max, \tag{3}$$

where the function $\theta(x, t_f)$ is defined by Eq. (1) under the constraints

$$\int_0^{t_f} q(\tau)\, d\tau = E \doteq \text{const}; \tag{4}$$

$$u_{\min} \leqslant q(t) \leqslant u_{\max} \tag{5}$$

(t_f is the duration of the process).

Solution. To reduce problem (3)–(5) to the problem of optimal control, it is necessary to transform the functional (3). Using Eq. (1) and changing the order of integration in (3), one can transform this functional into the following form:

$$\Phi(q) = \int_0^1 \theta(x, t_f) dx = \int_0^{t_f} q(\tau)\Psi(\tau) d\tau \to \max\left[\Psi(\tau) = 2H \sum_{n=1}^{\infty} \frac{1 - \cos\beta_n}{\cos\beta_n \{\beta_n^2 + H^2 + H\}} \exp\{-\beta_n^2(t_f - \tau)\}\right]. \tag{6}$$

Problem (4)–(6) is the problem of optimal control. It can be solved with the use of the Pontryagin maximum principle (or the equivalent minimum principle in Lagrangian form) [8–10]. This results in the following condition for the optimal control parameter $\hat{q}(t)$

$$\hat{q}(t)[\lambda_1 - \Psi(t)] \to \min, \tag{7}$$

where λ_1 is the Lagrange multiplier. Condition (7) with constraints (4) and (5) makes it possible to determine the optimal control parameter with the use of the relations

$$\hat{q}(t) = u_{\min} \quad \text{for} \quad \lambda_1 - \Psi(t) > 0; \tag{8a}$$

$$\hat{q}(t) = u_{\max} \quad \text{for} \quad \lambda_1 - \Psi(t) < 0. \tag{8b}$$

Relations (8a) and (8b) include the Lagrange multiplier λ_1 whose value is not known in advance. To determine it numerically, Eq. (4) is also used.

Examples of the optimal intensity of heat generation in the fluid versus the time for various durations

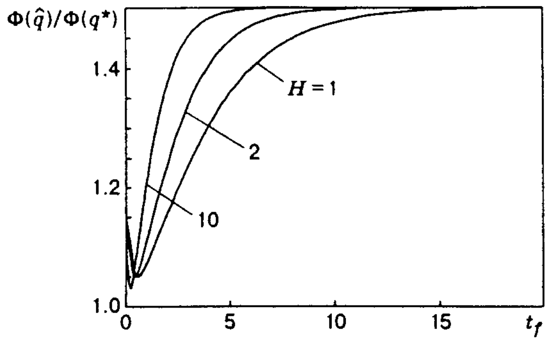

Fig. 2

of the process are given in Fig. 1. Calculations were performed for $u_{\min} = 1$, $u_{\max} = 3$, $H = 1$, and $E = 2t_f$. It is seen that when the duration is small ($t_f = 0.3$), the optimal intensity of heating first takes on the maximum admissible and then the minimum admissible value. When the duration of the process increases ($t_f = 0.6$). the behavior of the optimal intensity changes qualitatively. Now it first takes on the minimum admissible value, then the maximum admissible value, and again the minimum admissible value. As the duration further increases ($t_f = 2$), the behavior remains unchanged, but the duration of the third interval decreases. In the limit, as $t_f \to \infty$ the third interval vanishes, and the optimal intensity takes on the minimum value at the beginning of the process and then the maximum value.

To understand qualitatively the reason for such changes in the behavior of the optimal intensity of heating, we consider two limiting cases: the duration of the process is either very short or very long. When the duration is short, there is no loss of heat through the boundary $x' = 0$, and the first (maximum–minimum) type of behavior is advantageous. This is due to the fact that, in this case, the time of contact of the plate with the "hot" fluid is maximum. On the contrary, when the duration is long, the maximum intensity of heat generation should occur at the end of the process. Otherwise, almost all heat would be lost in passage through the boundary $x' = 0$. For the mean duration of the process, the maximum generation intensity should occur somewhere between the beginning and the end of the process, which causes the minimum–maximum–minimum type of behavior.

It is interesting to compare the values of the functional $\Phi(q)$ of the optimal function $\hat{q}(t)$ and of the function

$$q^*(t) = E/t_f = \text{const}. \tag{9}$$

The function $q^*(t)$ corresponds to a constant intensity of heat generation during the process and, clearly, satisfies the constraint (4). The ratio $\Phi(\hat{q})/\Phi(q^*)$ is the gain in energy which is obtained when the optimal intensity of heat generation in the fluid is used instead of the constant intensity of heat generation. Figure 2 shows this gain versus the process duration for various values of the parameter H that characterizes the ratio of the heat-accumulating capacities of the plate and the fluid. Calculations were carried out for $u_{\min} = 1$, $u_{\max} = 3$, and $E = 2T_f$. It is seen that the ratio $\Phi(\hat{q})/\Phi(q^*)$ first decreases and then increases to an asymptotic value which does not depend on the parameter H. This result makes it possible to hypothesize that the ratio $\Phi(\hat{q})/\Phi(q^*)$ tends to an asymptotic value as $t_f \to \infty$ for any regime such that the amount of heat E which can be generated during the process is proportional to its duration t_f.

We shall find this asymptotic value in the general formulation. Let the upper and lower boundaries of the interval of admissible control parameters be, respectively, $u_{\min}$ and $u_{\max}$. The minimum and maximum values of E are then determined by the expressions $E_{\min} = u_{\min} t_f$ and $E_{\max} = u_{\max} t_f$. We assume that the parameter E is proportional to t_f:

$$E = E_{\min} + \frac{E_{\max} - E_{\min}}{\omega} = t_f\left[u_{\min} + \frac{u_{\max} - u_{\min}}{\omega}\right] \quad (\omega > 1). \tag{10}$$

In accordance with Eqs. (6), (9) and (10), we have

$$\lim_{t_f\to\infty}\Phi(q^*) = \lim_{t_f\to\infty}\int_0^{t_f}\left[u_{\min}+\frac{u_{\max}-u_{\min}}{\omega}\right]2H\sum_{n=1}^{\infty}\frac{1-\cos\beta_n}{\cos\beta_n\{\beta_n^2+H^2+H\}}\exp\{-\beta_n^2(t_f-\tau)\}\,d\tau$$

$$= \left[u_{\min}+\frac{u_{\max}-u_{\min}}{\omega}\right]2H\sum_{n=1}^{\infty}\frac{1-\cos\beta_n}{\beta_n^2\cos\beta_n\{\beta_n^2+H^2+H\}}. \quad (11)$$

On the other hand, it follows from the analysis of Fig. 1 that, for $t_f \to \infty$, the optimal intensity of heat generation in the fluid can be represented in the form

$$\hat{q}(t) = u_{\min} \quad \text{for} \quad t < t_f\frac{\omega-1}{\omega}; \quad (12a)$$

$$\hat{q}(t) = u_{\max} \quad \text{for} \quad t > t_f\frac{\omega-1}{\omega}. \quad (12b)$$

The function $\hat{q}(t)$, which is determined by Eqs. (12a) and (12b), clearly satisfies the constraint (4) if the parameter E is determined by Eq. (10).

From Eqs. (6), (12a), and (12b), it follows that

$$\lim_{t_f\to\infty}\Phi(\hat{q}) = \lim_{t_f\to\infty}\Bigg\{\int_0^{t_f(\omega-1)/\omega} u_{\min}2H\sum_{n=1}^{\infty}\frac{1-\cos\beta_n}{\cos\beta_n\{\beta_n^2+H^2+H\}}\exp\{-\beta_n^2(t_f-\tau)\}\,d\tau$$

$$+\int_{t_f(\omega-1)/\omega}^{t_f} u_{\max}2H\sum_{n=1}^{\infty}\frac{1-\cos\beta_n}{\cos\beta_n\{\beta_n^2+H^2+H\}}\exp\{-\beta_n^2(t_f-\tau)\}\,d\tau\Bigg\} = u_{\max}2H\sum_{n=1}^{\infty}\frac{1-\cos\beta_n}{\beta_n^2\cos\beta_n\{\beta_n^2+H^2+H\}}, \quad (13)$$

and, from Eqs. (11) and (13), it follows that

$$\lim_{t_f\to\infty}\frac{\Phi(\hat{q})}{\Phi(q^*)} = q_{\max}\Big/\left(q_{\min}+\frac{q_{\max}-q_{\min}}{\omega}\right). \quad (14)$$

For example, for $u_{\min} = 1$, $u_{\max} = 3$, and $\omega = 2$ [according to Eq. (10), this results in the relation $E = 2T_f$], Eq. (14) yields

$$\lim_{t_f\to\infty}\frac{\Phi(\hat{q})}{\Phi(q^*)} = 1.5,$$

which coincides with the result obtained in Fig. 2. For the parameters of Fig. 2, the accumulated amount of thermal energy by the optimal intensity of heat generation is thus 1.5 times greater than by the constant intensity.

Conclusions. In the problem considered, it has been found that there exist two qualitatively different types of behavior of the optimal intensity of heat generation. When the duration of the process is short, the optimal generation intensity first takes the maximum admissible value and then the minimum admissible value. As the duration increases, the behavior of the optimal intensity changes. Now it first takes on the minimum admissible, then the maximum admissible, and again the minimum admissible value. The relative duration of the third interval decreases with increasing duration of the process. In the limit, when the duration tends to infinity, the optimal generation intensity first takes on the minimum admissible and then the maximum admissible value.

It has also been found that, in the case of a prolonged process, the gain in energy obtained at the optimal intensity of heat generation tends to an asymptotic value, which is not dependent on the ratio of the heat-accumulating capacities of the plate and the fluid.

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